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A NOTE ON THE VERTEX-DISTINGUISHING EDGE COLORING OF $P_m \vee K_n$ AND $C_m \vee K_n$

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Abstract

In this paper, we obtain the Vertex-distinguishing Edge Chromatic Number of $P_m \vee K_n$ and $C_m \vee K_n$.

1. Introduction

The problem which is due to computer science [1-6] about Vertex-distinguishing Edge Coloring of G is a widely applicable and extremely difficult problem. In [7] introduced the Vertex-distinguishing Edge Coloring of graph, and give the relevant conjecture.

Definition 1 [8-10]**.** G is a simple graph and k is a positive integer, if it exists a mapping of f, and satisfied with $f(e) \neq f(e')$ for adjacent edge $e, e' \in E(G)$, then f is called a *Proper Edge Coloring* of G, is abbreviated k-PEC of G, and

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$$\chi'(G) = \min\{k \mid k\text{-PEC}\}$$

is called the *Edge Chromatic Number* of G.

Definition 2 [1-6]**.** For the proper edge coloring f of simple graph, if it is satisfied with $C(u) \neq C(v)$ for $V(G)(u \neq v)$, where $C(U) = \{f(uv) \mid uv \in E(G)\}$, then f is called the *Vertex-distinguishing Edge Coloring*, is abbreviated k-VDEC of G, and

$$\chi'_{vd} = \min\{k \mid k\text{-VDEC}\}$$

is called the *Vertex-distinguishing Edge Chromatic Number* of G.

Definition 3. For a graph G, let n_i be the vertex number of the vertices of degree i, we call

$$\mu(G) = \max\left\{\min\left\{\lambda \mid \binom{\lambda}{i} \geq n_i,\ \delta \leq i \leq \Delta\right\}\right\}$$

the *Combinatorial Degree* of G, where δ and Δ are the minimal and maximal degree of G, respectively.

Conjecture [1-5]**.** For a connected graph G of order not less than 3, then

$$\mu(G) \leq \chi'_{vd}$$
$$\leq \mu(G) + 1.$$

Note that the left side of the inequality is obviously true.

Let G and H are two simple graphs, the joint graph of G and H, denote by $G \vee H$, is obtained from the disjoint union of G and H by making all of $V(G)$ adjacent to all of $V(H)$.

Because $P_1 \vee K_n = K_{n+1}$ and $P_2 \vee K_n = K_{n+2}$ has been discussed in another paper, we will consider the general case $P_m \vee K_n$ and $C_m \vee K_n$. The terms and signs we use in this paper but not denoted can be found in [8-10].

Lemma 1. *Let* $m \geq 3$ *and* $n \geq 4$, $\mu(P_m \vee K_n) = m + n$.

Proof. For $m = 3$ and $n = 3$, we can compute that

$$\max\left\{\min\left\{\theta \mid \binom{\theta}{6} \geq 2\right\} \text{ and } \min\left\{\theta \mid \binom{\theta}{7} \geq 6\right\}\right\} = 8.$$

For $n \geq 4$ and $m + n \neq 8$, we get that

$$\max\left\{\min\left\{\theta \mid \binom{\theta}{n+1} \geq 2\right\}, \min\left\{\theta \mid \binom{\theta}{n+2} \geq m-2\right\} \text{ and } \min\left\{\theta \mid \binom{\theta}{m+n-1} \geq n\right\}\right\}$$

$= m + n.$

Hence, the proof is finished.

Lemma 2 [5]. *For a complete graph* K_n, *then*

$$\chi'_{vd}(K_n) = \begin{cases} n+1, & \text{for } n \equiv 0 (\text{mod } 2); \\ n, & \text{for } n \equiv 1 (\text{mod } 2). \end{cases}$$

Lemma 3. *If* $m \geq 3$ *and* $n \geq 4$, *then*

$$\mu(C_m \vee K_n) = m + n.$$

Proof. We have that

$$\mu(C_m \vee K_n)$$

$$= \max\left\{\min\left\{\theta \mid \binom{\theta}{n+2} \geq m\right\} \text{ and } \min\left\{\theta \mid \binom{\theta}{m+n-1} \geq n\right\}\right\}$$

$$= m + n.$$

2. Results about $P_m \vee K_n$

Theorem 2.1. *If* $m + n \neq 3$, *then*

$$\chi'_{vd}(P_m \vee K_n) = \begin{cases} n+1, & m = 1, n = 0(\text{mod } 2); \\ n+2, & m = 1, n = 1(\text{mod } 2); \quad m = 2, n = 1(\text{mod } 2); \\ n+3, & m = 1, n = 0(\text{mod } 2). \end{cases}$$

Proof. When $m = 1, 2,$ we can get $P_m \vee K_n = K_{m+n}$ from [5], the conclusion is true.

Theorem 2.2. *If* $m \geq 3$ *and* $n \geq 4$, *then*

$$\chi'_{vd}(P_m \vee K_n) = m + n.$$

Proof. Let the path $P_m = u_1 u_2 \cdots u_m$ and $V(K) = \{u_{m+1}, u_{m+2}, \ldots, u_{m+n}\}$ and $C = \{1, 2, \ldots, m+n-1, 0\}$. From Lemma 2, we only need to prove that there exists a $(m+n)$-VDEC of $P_m \vee K_n$. Hence, we can make a proper edge coloring f of $P_m \vee K_n$ as:

$$f(u_i u_j) = i + j - 1 (\text{mod } m+n) \quad \text{for } 1 \le i \le n \quad \text{and} \quad m+1 \le j \le m+n,$$

and

$$f(u_{m+i} u_{m+j}) = 2m + i + j - 2 (\text{mod } m+n) \quad \text{for} \quad i \le i, \quad j \le n.$$

Let the color subtractive set $\overline{C}(u) = C \backslash C(u)$ for $u \in V(P_m \vee K_n)$.

Case 1. If $m > n \ge 4$, $f(u_i u_{i+1}) = i$ for $1 \le i \le n$; we can compute that

$\overline{C}(v_i) = \{2(i-1)\}$, for $1 \le i \le n$;

$C(u_1) = \{1, n, n+1, \ldots, 2n-1\}$;

$C(u_m) = \{m-1, m+n-1, 0, \ldots, n-2\}$;

$C(u_i) = \{i-1, i, n+i-1, n+i, \ldots, 2n+i-2\} (\text{mod } m+n)$, for $2 \le i \le m-1$.

Thus f is a $(m+n)$-VDEC of $P_m \vee K_n$. This proves that the result is true.

Case 2. If $m = n$, $f(u_i u_{i+1}) = i$ for $1 \le i \le n-1$, there are

$\overline{C}(v_i) = \{2(i-1)\}$, for $1 \le i \le n$;

$C(u_1) = \{1, n, n+1, \ldots, 2n-1\}$;

$C(u_m) = \{n-1, 2n-1, 0, \ldots, n-2\}$;

$C(u_i) = \{i-1, i, n+i-1, n+i, \ldots, 2n+i-2\} (\text{mod } 2n)$, for $2 \le i \le n-1$.

Hence, f is $(m+n)$-VDEC of $P_m \vee K_n$.

Case 3. If $n > m$, there are

$$f(u_i u_{i+1}) = n - m + i, \text{ for } 1 \le i \le m-1,$$

we get that

$$\overline{C}(v_i) = \{2(i-1)\}, \quad \text{for } 1 \le i \le \frac{m+n}{2};$$

$$\overline{C}(v_i) = \{2i - m - n + 1\}, \quad \text{for } \frac{m+n}{2} \le i \le n.$$

For $m + n \equiv 1 (\text{mod } 2)$, there have

$$\overline{C}(v_i) = \{2(i-1)\}, \quad \text{for } 1 \le i \le \frac{m+n+1}{2};$$

$$\overline{C}(v_i) = \{2i - m - n + 1\}, \quad \text{for } \frac{m+n+1}{2} + 1 \le i \le n,$$

$$C(u_1) = \{n - m + 1, n, n + 1, \dots, 2n - 1\} (\text{mod } m + n);$$

$$C(u_m) = \{n - 1, m + n - 1, 0, 1, \dots, n - 2\};$$

$$C(u_i) = \{n - m + i, n - m + i + 1, n + i - 1, \dots, 2n + i - 2\} (\text{mod } m + n),$$

for $1 \le i \le m - 1$.

Therefore, f is a $(m + n)$-VDEC of $P_m \vee K_n$. The proof is finished.

3. Results about $C_m \vee K_n$

Theorem 3.1. *If* $n > 1$, *then*

$$\chi'_{vd}(C_3 \vee K_n) = \begin{cases} n + 4, & n = 1(\text{mod } 2); \\ n + 3, & n = 0(\text{mod } 2). \end{cases}$$

Proof. Because of $C_3 \vee K_n = K_{n+3}$, the result is true we know by [5].

Theorem 3.2. *If* $m \ge 4$ *and* $n \ge 4$, *then*

$$\chi'_{vd}(C_m \vee K_n) = m + n.$$

Proof. By Lemma 2, the inequality $\chi'_{vd}(C_3 \vee K_n) \ge \mu(C_3 \vee K_n)$ is obvious, so we only need to prove that $C_3 \vee K_n$ has a mapping $(m+n)$-VDEC only. For convenient, we let that

$$C_m = u_1 u_2 \cdots u_m u_1,$$

$$V(K_n) = \{v_i \mid i = 1, 2, \ldots, n\};$$

$$C = \{1, 2, \ldots, m+n-1, 0\},$$

$$\overline{C}(v) = C \backslash C(v),$$

$$u_i = v_{n+i}, \quad i = 1, 2, \ldots, m.$$

Case 1. If $m > n$, we make a coloring function *f* as:

$$f(v_i v_j) = i + j - 2(\operatorname{mod} m+n),$$

for $i = 1, 2, \ldots, n;$ $j = i+1, i+2, \ldots, m+n$ and $f(u_i u_{i+1}) = i,$ $i = 1, 2, \ldots, m-1;$ and $f(u_m u_1) = n-1.$

Therefore, we can get that

$\overline{C}(v_i) = \{2(i-1)\},$ for $1 \le i \le n;$

$C(u_1) = \{1, n-1, n, \ldots, 2n-1\};$

$C(u_m) = \{m-1, m+n-1, 0, 1, \ldots, n-1\};$

$C(u_i) = \{i-1, i, n+i-1, \ldots, 2n+i-2\}(\operatorname{mod} m+n),$ for $2 \le i \le m-1.$

This proves that *f* is a $(m+n)$-VDEC of $C_m \vee K_n$.

Case 2. If $m = n$, we make *f* as:

$$f(v_i v_j) = i + j - 2(\operatorname{mod} 2n), \quad i = 1, 2, \ldots, n; \quad j = i+1, i+2, \ldots, 2n$$

and

$$f(u_i u_{i+1}) = i+1, \quad i = 1, 2, \ldots, n-1;$$

and $f(u_n u_1) = n-1.$

Then, we still have that

$\overline{C}(v_i) = \{2(i-1)\}, \quad i = 1, 2, \ldots, n;$

$C(u_1) = \{2, n-1, n, \ldots, 2n-1\};$

$C(u_m) = \{n-1, n, 2n-1, 0, 1, \ldots, n-2\};$

$C(u_i) = \{i, i+1, n+i-1, \ldots, 2n+i-2\}(\text{mod } 2n), \quad i = 2, 3, \ldots, n-1.$

That means that *f* is a $(2n)$-VDES of $C_m \vee K_n$.

Case 3. If $n > m$, we let *f* as:

$$f(v_i v_j) = i + j - 2(\text{mod } m+n), \quad i = 1, 2, \ldots, n; \quad j = i+1, i+2, \ldots, m+n$$

and

$$f(u_i u_{i+1}) = n - m + i, \quad i = 1, 2, \ldots, m-2;$$

and $f(u_{m-1}u_m) = n$ and $f(u_m u_1) = n-1$.

Then, if $m + n \equiv 0(\text{mod } 2)$, we can see that

$\overline{C}(v_i) = \{2(i-1)\}, \quad i = 1, 2, \ldots, \frac{m+n}{2};$

$\overline{C}(v_i) = \{2i-(m+n)-1\}, \quad i = \frac{m+n}{2}+1, \frac{m+n}{2}+2, \ldots, n;$

$C(u_1) = \{n-m+1, n-1, n, \ldots, n-m-1\};$

$C(u_{m-1}) = \{n-2, n, m+n-2, m+n-1, 0, 1, \ldots, n-3\};$

$C(u_m) = \{n-1, n, m+n-1, 0, \ldots, n-2\};$

and

$$C(u_i)$$

$$= \{n-m+i, n-m+i+1, n+i-1, \ldots, n-m+i-2\}(\text{mod } n+m), \quad i = 2, 3, \ldots, m-2.$$

If $m + n \equiv 1 (\bmod 2)$, we can compute

$$\overline{C}(v_i) = \{2(i-1)\}, \quad i = 1, 2, \ldots, \frac{m+n+1}{2};$$

$$\overline{C}(v_i) = \{2i - m - n\}, \quad i = \frac{m+n+1}{2} + 1, \frac{m+n+1}{2} + 2, \ldots, n;$$

$$C(u_1) = \{n-1, n-m+1, n, n+1, \ldots, n-m-1\};$$

$$C(u_{m-1}) = \{n-2, n, m+n-2, m+n-1, 0, 1, \ldots, n-3\};$$

$$C(u_m) = \{n-1, n, m+n-1, 0, \ldots, n-2\};$$

$$C(u_i)$$

$$= \{n-m+i-1, n-m+i, n+i-1, \ldots, n-m+i-2\} (\bmod n+m), \quad i = 2, 3, \ldots, m-2.$$

We have proved that f is a $(m+n)$-VDEC of $C_m \vee K_n$.

The proof is completed.

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