

JENSEN AND HERMITE-HADAMARD INCLUSIONS FOR STRONGLY CONCAVE SET-VALUED MAPS

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Abstract

Counterparts of the classical integral and discrete Jensen inequalities and the Hermite-Hadamard Theorem and its converse, for strongly concave set-valued maps, are presented.

1. Introduction

Let \( I \subset \mathbb{R} \) be an interval and let \( c \) be a positive number. A function \( f : I \to \mathbb{R} \) is called strongly convex with modulus \( c \) if

\[
    f((1-t)x + ty) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2,
\]

(1)

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for all \( x, y \in I \) and \( t \in (0, 1) \); \( f \) is called strongly concave with modulus \( c \) if \( -f \) is strongly convex with modulus \( c \). Strongly convex functions have been introduced by Polyak [16] and they play an important role in optimization theory and mathematical economics. Many properties and applications of them can be found in [9], [13], [17] and [23].

Recently, Huang [6] extended the definition (1) of strongly convex functions to set-valued maps. Some properties and applications of them, can be found in [4], [7] and [15].

Roughly speaking, in the case of set-valued maps convexity and concavity are different to the case of real-valued functions, in view of the fact that if \( F \) is strongly convex set-valued map, then \( -F \) is also strongly convex and although some properties of strongly convex and strongly concave set-valued maps are similar, they hold, in general, under different assumptions and have to be proved separately. Recently, strongly concave set-valued maps were introduced in [11] and the authors exhibited some properties of them. Our paper is related to [15] where analogous results for strongly convex set-valued maps are presented.

2. Preliminaries

Throughout this paper \( Y \) is a Banach space, \( B \) is the closed unit ball in \( Y \), \( I \subset \mathbb{R} \) denotes an interval and \( c \) is a positive constant.

Denote by \( n(Y) \) the family of all nonempty subset of \( Y \), and by \( \text{conv}(Y) \) and \( c\text{conv}(Y) \) the subfamilies of \( n(Y) \) of all convex and compact convex sets, respectively.

A set-valued map \( F : I \to n(Y) \) is called strongly concave with modulus \( c \), if

\[
F(tx + (1-t)y) + cy(1-t)(x-y)^2 B \subseteq tF(x) + (1-t)F(y),
\]

(2)

for all \( x, y \in I \) and \( t \in (0, 1) \) (see [11]).

\( F \) is called concave if it satisfies (2) with \( c = 0 \) (see, for instance [12], [14], [18]).
A function $f : I \to Y$ is called a selection of $F : I \to n(Y)$ if $f(x) \in F(x)$ for every $x \in I$.

The following lemma characterizes strongly concave set-valued maps with values in $cconv(\mathbb{R})$ and shows connections between conditions (1) and (2) (see [15] where an analogous result for strongly convex set-valued maps is given).

**Lemma 2.1.** If a set-valued map $F : I \to cconv(\mathbb{R})$ is strongly concave with modulus $c$, then it has the following form $F(x) = [f_1(x), f_2(x)]$, where $f_1, f_2 : I \to \mathbb{R}, f_1 \leq f_2$ on $I$, $f_1$ is strongly concave with modulus $c$ and $f_2$ is strongly convex with modulus $c$.

Conversely, if $f_1, f_2 : I \to \mathbb{R}$, where $f_1$ is strongly concave with modulus $c$ and $f_2$ is strongly convex with modulus $c$, $f_1 \leq f_2$ on $I$, then, the set-valued maps $F_1, F_2, F_3$ and $F_4$ defined by $F_1(x) = [f_1(x), f_2(x)], F_2(x) = [f_1(x), +\infty), F_3(x) = (\infty, f_2(x)], F_4(x) = (\infty, +\infty), x \in I$, are strongly concave with modulus $c$.

**Proof.** Assume first that $F : I \to cconv(\mathbb{R})$ is strongly concave with modulus $c$. Then, for every $x \in I$, $F(x)$ is a closed and bounded interval of $\mathbb{R}$ thus, $F(x) = [f_1(x), f_2(x)], \forall x \in I$.

In order to prove that $f_1(x) = \inf F(x)$ is strongly concave with modulus $c$, notice that by (2), we have

$$F(tx + (1 - y)y) + ct(1 - t)(x - y)^2[-1, 1] \subset tF(x) + (1 - t)F(y),$$

for all $x, y \in I$ and $t \in (0, 1)$.

Thus

$$\inf(tF(x) + (1 - t)F(y)) \leq \inf(F(tx + (1 - t)y) + ct(1 - t)(x - y)^2[-1, 1])$$

$$\leq \inf F(tx + (1 - t)y) - ct(1 - t)(x - y)^2$$
and, consequently
\[ tf_1(x) + (1 - t) f_1(y) + ct(1 - t)(x - y)^2 \leq f_1(tx + (1 - t)y), \]
which shows that \( f_1 \) is strongly concave with modulus \( c \). The fact that \( f_2(x) = \sup F(x) \) is strongly convex, with modulus \( c \), is proved similarly.

This finishes the first part of proof.

Now, suppose that \( f_1, f_2 : I \rightarrow R \), where \( f_1 \) is strongly concave with modulus \( c \) and \( f_2 \) is strongly convex with modulus \( c \). We shall show that \( F_2(x) = [f_1(x), +\infty) \) is strongly concave with modulus \( c \) (the proofs in the remaining cases are similar). Indeed,
\[
F_2(tx + (1 - t)y) + ct(1 - t)(x - y)^2[-1, 1]
\]
\[
= [f_1(tx + (1 - t)y) - ct(1 - t)(x - y)^2, +\infty]
\]
\[
\subseteq [tf_1(x) + (1 - t)f_2(y), +\infty]
\]
\[
= tF_2(x) + (1 - t)F_2(y),
\]
which finishes the proof. \( \square \)

3. The Jensen Inclusion

In [9], the following version of the Jensen inequality was proved: If \( f : I \rightarrow \mathbb{R} \) is strongly convex with modulus \( c \), then
\[
f\left( \int_X \varphi(w) d\mu \right) \leq \int_X f(\varphi(w)) d\mu - c \int_X (\varphi(w) - m)^2 d\mu,
\]
where \( m = \int_X \varphi(w) d\mu \), for every probability measure space \((X, \Sigma, \mu)\) and each
\( \mu \)-integrable function \( \varphi : X \rightarrow I \). From this result it readily follows that if \( f \) is strongly concave with modulus \( c \), then

\[
f \left( \int_X \varphi(w)\,d\mu \right) \geq \int_X f(\varphi(w))\,d\mu + c \int_X (\varphi(w) - m)^2\,d\mu. \tag{4}
\]

A counterpart of (3) for set-valued maps was obtained in [15]. In the next theorem, we give a counterpart of (4) for strongly concave set-valued maps. We consider the integral of a set-valued map in the sense of Aumann; that is, as the set of all the integrals of all \( \mu \)-integrable selections of it.

**Theorem 3.1.** Let \( (X, \Sigma, \mu) \) be a probability measure space. If \( F : I \rightarrow \text{cconv}(Y) \) is strongly concave with modulus \( c \), then for each square-integrable function \( \varphi : X \rightarrow I \),

\[
f \left( \int_X \varphi(w)\,d\mu \right) + c \int_X (\varphi(w) - m)^2\,d\mu \subset \int_X F(\varphi(w))\,d\mu, \tag{5}
\]

where \( m = \int_X \varphi(w)\,d\mu \).

**Proof.** The proof is divided into two steps.

First, we assume that \( Y = \mathbb{R} \), then, by Lemma 2.1, \( F \) has the form \( F(x) = \| f_1(x), f_2(x) \| \), \( x \in I \), with \( f_1 \) is strongly concave with modulus \( c \) and \( f_2 \) strongly convex with modulus \( c \).

Put \( m := \int_X \varphi(w)\,d\mu \in I \) and consider an arbitrary \( y \in F(m) = \| f_1(m), f_2(m) \| \).

Then, using Jensen inequalities (3) and (4) we have

\[
\int_X f_1(\varphi(w))\,d\mu + c \int_X (\varphi(w) - m)^2\,d\mu \leq f_1 \left( \int_X \varphi(w)\,d\mu \right)
\]
\[
\int_X f_2(\varphi(w)) \, d\mu \leq \int_X (\varphi(w) - m)^2 \, d\mu.
\]
whence
\[
\int_X f_1(\varphi(w)) \, d\mu \leq \int_X (\varphi(w) - m)^2 \, d\mu,
\]
\[
\leq y - c \int_X (\varphi(w) - m)^2 \, d\mu,
\]
\[
\leq y + c \int_X (\varphi(w) - m)^2 \, d\mu,
\]
\[
\leq \int_X f_2(\varphi(w)) \, d\mu.
\]
Thus
\[
y + c \int_X (\varphi(w) - m)^2 \, d\mu [-1, 1] \subseteq \left[ \int_X f_1(\varphi(w)) \, d\mu, \int_X f_2(\varphi(w)) \, d\mu \right]
\]
\[
\subseteq \int_X F(\varphi(w)) \, d\mu
\]
and consequently
\[
F\left( \int_X \varphi(w) \, d\mu \right) + c \int_X (\varphi(w) - m)^2 \, d\mu [-1, 1] \subseteq \int_X F(\varphi(w)) \, d\mu,
\]
which finishes the proof in the case \( Y = \mathbb{R} \).

Now assume that \( Y \) is an arbitrary Banach space. Take a nonzero continuous linear functional \( y^* \in Y^* \) and consider the set-valued map \( G : I \to \text{conv}(\mathbb{R}) \) defined by \( G(x) := (y^* o F)(x) \). This set-valued map is well defined, it is strongly concave with modulus \( c\|y^*\| \) and has compact convex values in \( \mathbb{R} \). Therefore, by
the previous step
\[
G\left(\int_X \varphi(w) d\mu\right) + c\left\|y^*\right\| \int_X (\varphi(w) - m)^2 d\mu[-1, 1] \subseteq \int_X G(\varphi(w)) d\mu; \quad (6)
\]
that is
\[
(y^* o F)\left(\int_X \varphi(w) d\mu\right) + c\left\|y^*\right\| \int_X (\varphi(w) - m)^2 d\mu[-1, 1] \subseteq \int_X (y^* o F)(\varphi(w)) d\mu. \quad (6')
\]

Fix a point \(b \in B\) and take an arbitrary \(y \in \left(\int_X \varphi(w) d\mu\right) = F(m)\), then
\[
y + c \int_X (\varphi(w) - m)^2 d\mu b \in F\left(\int_X \varphi(w) d\mu\right) + c \int_X (\mu(w) - m)^2 d\mu B.
\]
Hence, by \(6'\),
\[
y^*\left(y + c \int_X (\varphi(w) - m)^2 d\mu b\right) \in y^* (y) + c\left\|y^*\right\| \int_X (\varphi(w) - m)^2 d\mu[-1, 1]
\]
\[
\subseteq \int_X (y^* o F)(\varphi(w)) d\mu
\]
\[
\subseteq y^* \int_X F(\varphi(w)) d\mu.
\]

Since this condition holds for arbitrary \(y^* \in Y^*\) and the set \(y^* \left(\int_X F(\varphi(w)) d\mu\right)\) is convex and closed, by the separation theorem (see [20, Corollary 2.5.11]) we obtain
\[
y + c \int_X (\varphi(w) - m)^2 d\mu b \in \int_X F(\varphi(w)) d\mu
\]
and since \(y\) and \(b\) arbitraries, we conclude that
\[
F\left(\int_X \varphi(w) d\mu\right) + c \int_X (\varphi(w) - m)^2 d\mu B \subseteq \int_X F(\varphi(w)) d\mu. \quad \square
\]
As a consequence of Theorem 3.1, we obtain the following discrete Jensen inclusion for strongly concave set-valued maps.

**Corollary 3.1.** If $F : I \to \text{econv}(Y)$ is strongly concave with modulus $c$, then

\[
F\left(\sum_{i=1}^{n} t_i x_i\right) + c \sum_{i=1}^{n} t_i (x_i - m)^2 B \subseteq \sum_{i=1}^{n} t_i F(x_i)
\]

for all $n \in \mathbb{N}$, $x_1, x_2, \ldots, x_n \in I$, $t_1, \ldots, t_n > 0$ with $\sum_{i=1}^{n} t_i = 1$ and $m = \sum_{i=1}^{n} t_i x_i$.

**Proof.** Suppose that $X = I$, $\varphi(x) = x$ for $x \in I$, and that $x_1, x_2, \ldots, x_n \in I$ are distinct points. Moreover, assume that $\mu$ is a probability measure concentrated at $x_1, x_2, \ldots, x_n$, that is, $\mu(x_i) = t_i > 0$, $i = 1, \ldots, n$ and $\sum_{i=1}^{n} t_i = 1$. Then

\[
m = \int_X \varphi(w) \, d\mu = \sum_{i=1}^{n} t_i x_i, \quad \int_X (\mu(w) - m)^2 \, d\mu = \sum_{i=1}^{n} t_i (x_i - m)^2
\]

and

\[
\int_X F(\varphi(w)) \, d\mu = \sum_{i=1}^{n} t_i F(x_i).
\]

Now, using the strong concavity of $F$ and substituting the above equalities in (5), we get

\[
F\left(\sum_{i=1}^{n} t_i x_i\right) + c \sum_{i=1}^{n} t_i (x_i - m)^2 B \subseteq \sum_{i=1}^{n} t_i F(x_i),
\]

which finishes the proof. $\square$

### 4. The Hermite-Hadamard Inclusion

It is know that if a function $f : I \to \mathbb{R}$ is strongly convex with modulus $c$, then
it satisfies the version of the Hermite-Hadamard double inequality
\[
\int_a^b f(x) dx \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2} - \frac{c}{6} (b-a)^2,
\] for all \( a, b, \in I \) with \( a < b \) (see [9]). In the case that \( f \) is convex then it satisfies (7) with \( c = 0 \).

It readily follows that if \( f \) is strongly concave with modulus \( c \), then the version of the Hermite-Hadamard has the form
\[
f\left(\frac{a+b}{2}\right) + \frac{c}{12} (b-a)^2 \geq \frac{1}{b-a} \int_a^b f(x) dx \geq \frac{f(a) + f(b)}{2} + \frac{c}{6} (b-a)^2.
\] (8)

Recently, the authors in [15] obtained a counterpart of the above inequality (7) for strongly convex set-valued maps. In this section, we present a counterpart of (8) for strongly concave set-valued maps.

**Theorem 4.1.** If a set-valued map \( F : I \to c\text{conv}(Y) \) is strongly concave with modulus \( c \), then
\[
F\left(\frac{a+b}{2}\right) + \frac{c}{12} (b-a)^2 B \subseteq \frac{1}{b-a} \int_a^b F(x) dx, \quad a, b, \in I, \quad a < b
\] (9)

and
\[
\frac{1}{b-a} \int_a^b F(x) dx + \frac{c}{6} (b-a)^2 B \subseteq \frac{F(a) + F(b)}{2}, \quad a, b, \in I, \quad a < b.
\] (10)

**Proof.** To show that condition (9) holds, take \( X = [a, b] \), \( \varphi : [a, b] \to I \) defined by \( \varphi(x) = x \) and \( \mu = \frac{1}{b-a} \lambda \), where \( \lambda \) is the Lebesgue measure on \( \mathbb{R} \).

Then
\[
\int_X \varphi(x) d\mu = \frac{a + b}{2},
\]
\[ F\left( \int_X \varphi(x) \, d\mu \right) = F\left( \frac{a+b}{2} \right). \]

\[ \int_X (\varphi(x) - m)^2 \, d\mu = \frac{1}{12} (b-a)^2 \]

and

\[ \int_X F(\varphi(x)) \, d\mu = \frac{1}{b-a} \int_a^b F(x) \, dx. \]

Using the fact that \( F \) is strongly concave with modulus \( c \), by substituting these equalities in (5) we get (9).

To prove (10), consider \( a, b \in I, \ a < b, \ f : [a, b] \to Y \) a \( \mu \)-integrable selection of \( F \) and \( \rho \in \rho_B \).

Express \( F(x) = F\left( \frac{b-x}{b-a} a + \frac{x-a}{b-a} b \right), \ x \in [a, b], \) and take

\[ f(x) + c\left( \frac{b-x}{b-a} \right) \left( \frac{x-a}{b-a} \right) (b-a)^2 \rho \]

\[ \in F\left( \frac{b-x}{b-a} a + \frac{x-a}{b-a} b \right) + c\left( \frac{b-x}{b-a} \right) \left( \frac{x-a}{b-a} \right) (b-a)^2 B. \]

By the strong concavity of \( F \), it follows that

\[ f(x) + c\left( \frac{b-x}{b-a} \right) \left( \frac{x-a}{b-a} \right) (b-a)^2 \rho \in \frac{b-x}{b-a} F(a) + \frac{x-a}{b-a} F(b), \]

hence, there exist \( u \in F(a) \) and \( v \in F(b) \) such that

\[ f(x) + c\left( \frac{b-x}{b-a} \right) \left( \frac{x-a}{b-a} \right) (b-a)^2 \rho = \frac{b-x}{b-a} u + \frac{x-a}{b-a} v. \]
On the other hand, by simple calculations we have
\[
\frac{1}{b-a} \int_a^b f(x) \, dx + \frac{c}{6} (b-a)^2 \rho = \frac{u + v}{2} \in \frac{F(a) + F(b)}{2}.
\]

Therefore, we conclude that
\[
\frac{1}{b-a} \int_a^b F(x) \, dx + \frac{c}{6} (b-a)^2 B \subset \frac{F(a) + F(b)}{2}.
\]

5. The Converse of the Hermite-Hadamard Theorem

In [9], the following version for the converse of the Hermite-Hadamard Theorem was proved: if a continuous function \( f : I \to \mathbb{R} \) satisfies the left or the right hand-side inequality of (7), then it is strongly convex with modulus \( c \). If \( f \) satisfies the same conditions on (7) with \( c = 0 \), then \( f \) is convex.

Similarly, if \( f \) is continuous and satisfies the left or the right hand-side inequality of (8), then \( f \) is strongly concave with modulus \( c \).

In what follows we assume that \( Y \) is a separable Banach space and denote by \( bccl(Y) \) the family of all bounded convex closed and non-empty subsets of \( Y \).

Recently, in [15], it was proved a counterpart for the converse of the Hermite-Hadamard Theorem for strongly convex set-valued maps: if \( F : I \to bccl(Y) \) is continuous and satisfies
\[
\frac{1}{b-a} \int_a^b F(x) \, dx + \frac{c}{12} (a-b)^2 B \subset F\left(\frac{a+b}{2}\right), \quad a, b \in I, \quad a < b
\]
or
\[
\frac{F(a) + F(B)}{2} + \frac{c}{6} (a-b)^2 B \subset \frac{1}{b-a} \int_a^b F(x) \, dx, \quad a, b \in I, \quad a < b,
\]
then \( F \) is strongly convex with modulus \( c \).
In this section, we present the converse of the Hermite-Hadamard Theorem for strongly concave set-valued maps.

Recall that a set-valued map $F : I \rightarrow \text{conv}(Y)$ is said to be continuous at a point $x_0$ if for every neighborhood $V$ of zero in $Y$ there exists a neighborhood $U$ of zero in $R$ such that

$$F(x) \subset F(x_0) + V \quad \text{and} \quad F(x_0) \subset F(x) + V$$

for all $x \in (x_0 + U) \cap I$.

Now, we show the converse of the Hermite-Hadamard Theorem for strongly concave set-valued maps.

**Theorem 5.1.** If a set-valued map $F : I \rightarrow \text{conv}(Y)$ is continuous and satisfies

$$\frac{1}{b-a} \int_a^b F(x)dx + \frac{c}{6} (b-a)^2 B \subset \frac{F(a) + F(b)}{2}, \quad a, b \in I, \quad a < b, \quad (11)$$

or

$$F\left(\frac{a+b}{2}\right) + \frac{c}{12} (b-a)^2 B \subset \frac{1}{b-a} \int_a^b F(x)dx, \quad a, b \in I, \quad a < b, \quad (12)$$

then $F$ is strongly concave with modulus $c$.

**Proof.** The idea of the proof is taken from [10, Theorem 8].

Suppose that $F$ is not strongly concave with modulus $c$, i.e., (2) does not hold. Then, there are $t_0 \in (0, 1)$, $x_1, x_2 \in I$ and $z \in F(t_0 x_1 + (1-t_0)x_2)

+ ct_0(1-t_0)(x_1-x_2)^2 B$ such that $z \notin t_0 F(x_1) + (1-t_0)F(x_2)$.

Since the set $t_0 F(x_1) + (1-t_0)F(x_2)$ is convex and closed, by the separation theorem, there exists a continuous linear functional $y^* \in Y^*$ such that

$$y^*(z) > \sup\{y^*(y) : y \in t_0 F(x_1) + (1-t_0)F(x_2)\}. \quad (13)$$
Now, if $F$ satisfies (11) (the proof in case that $F$ satisfies (12) is similar), then

$$y^*\left(\frac{1}{b-a}\int_a^b F(x) \, dx + \frac{c}{6} (b-a)^2 B\right) \leq \frac{y^*(F(a)) + y^*(F(b))}{2}.$$  \hspace{1cm} (14)

Consider the function $f : I \to \mathbb{R}$ defined by $f(x) := \sup y^*(F(x))$, $x \in I$. Clearly, $f$ is continuous and, in view of (14) and the fact that

$$\int_a^b \sup y^*(F(x)) \, dx = \sup y^* \int_a^b F(x) \, dx$$

(see [5, Proposition 5.2]), it satisfies

$$\frac{1}{b-a} \int_a^b f(x) \, dx + \frac{c}{6} (b-a)^2 \leq \frac{f(a) + f(b)}{2},$$

that is, $f$ is continuous and satisfies the right hand-side inequality of (7). Thus by the converse of Hermite-Hadamard inequalities we have that $f$ is strongly convex with modulus $c$, in particular,

$$f(t_0x_1 + (1-t_0)x_2) + ct_0(1-t_0)(x_1 - x_2)^2 \leq t_0 f(x_1) + (1-t_0)f(x_2).$$

Consequently, by the definition of $f$, we obtain

$$y^*(z) \leq \sup y^*(F(t_0x_1 + (1-t_0)x_2) + ct_0(1-t_0)(x_1 - x_2)^2 B)$$

$$\leq t_0 \sup y^*(F(x_1)) + (1-t_0) \sup y^*(F(x_2))$$

$$= \sup y^*(t_0F(x_1) + (1-t_0)F(x_2)).$$

this contradicts (13) and finishes the proof. \hfill \Box

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