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HIGH-ORDER ACCURATE NUMERICAL SCHEME FOR NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

CHINEDU NWAIGWE[1], **AZUBUIKE WELI**[2] **and**
JAMES UBUO OKAFOR[3]

[1, 2]Department of Mathematics
Rivers State University
Nigeria
e-mail: ynwaigwe.chinedu@ust.edu.ng

[3]Department of Mathematics
Abia State Polytechnic
Aba, Nigeria

Abstract

A fourth-order numerical method is proposed for solving first-order nonlinear integro-differential equations. The method is based on finite difference approximation of derivatives and an unconventional quadrature approximation of integrals. The unconventional quadrature scheme emanates from approximating the leading error term of the conventional trapezoid quadrature rule. The nonlinear non-homogeneous parts are discretely interpolated. This resulted to a fully nonlinear algebraic system which is approximated using a nonlinear solver. The proposed method is tested on two nonlinear non-homogeneous equations with known exact solutions. Numerical solutions are observed for convergence, order of accuracy and appearance of non-physical oscillations. The results show that the method is convergent, has fourth-order

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of accuracy, and produce no non-physical oscillations for all mesh sizes - which is a numerical attestation of the stability of the proposed method.

1. Introduction

Many of the nonlinear phenomena that occur in industry and technology can be formulated as nonlinear integro-differential equations [1]. For instance, integro-differential equations arise in heat transfer with memory [2, 3], motion of viscoelastic materials [4], infectious disease modeling [5], optimal control [6], hematopoietic stem cell modeling [7], modeling complex interaction of active particles in mathematical kinetic theory [8], and in modeling tumor growth with effect of chemotherapy treatment [9].

The numerous applications mentioned above have elicited a lot of research on the theory and solution of integro-differential equations (IDEs).

The theoretical analysis of IDEs can be found in [1]. In the applications mentioned above the resulting IDEs are generally nonlinear hence finding their exact solution in closed form is generally not possible. Hence, numerical algorithms for IDEs have also seen a lot of research interests. Several numerical approaches have evolved for approximating the solution of this kind of equations. For instance, Al-Saar and Ghadle [10] suggested some methods based on the modification of the variational iteration methods, Laplace Adomain decomposition method and the Homotopy perturbation methods. Shirani, et al. [11] used shifted Lengendre polynomials to solve nonlinear Volterra-Fredholm integro-differential equations. In [12], Chebyshev pseudo-spectral method is used to approximate the solution of systems of Fredholm integro-differential equations. A differential-integral quadrature method is suggested for discretizing an integro-differential equation in [13]. Other existing methods include the Nystrom method [14], variational iteration methods [15] and the exponential spline approach [16]. The order of accuracy of these methods is hardly investigated nor reported and most of them cannot be easily implemented, even their derivations are very complicated, requiring so much calculations and computations.

Consequently, there is still a need for a method which is easy to formulate, yet has high order accuracy and is simple to automate. This is the thrust of this current paper - to propose, implement and verify a fourth-order convergent numerical method for the following nonlinear Fredholm integro-differential equation:

$$\frac{du}{dx} = g(x) + f(x, u(u)) + \int_a^b k(x, y, u(y))dy, \quad x \in (a, b) \subset \mathbb{R}, \tag{1}$$

$$u(a) = u_0,$$

where a and b are both constants; $g(x)$, $f(x, u(x))$ and $k(x, y, u(y))$ are known functions of their arguments. The function $k(x, y, u(y))$ is known as the *kernel of the integral*, $u(x)$ is the unknown function that is to be determined, $g : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, $k : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. We assume that function f is Lipschitz continuous with respect to the second argument and with a Lipschitz constant $a_1 \geq 0$,

$$| f(x, z_1) - f(x, z_2) | \leq a_1 | z_1 - z_2 | \quad \text{for all} \quad z_1, \; z_2.$$

We also assume that the kernel k is Lipschitz continuous with respect to the third argument, and with a Lipschitz constant $a_2 \geq 0$. With these conditions, it can be shown that the problem has a unique solution, see [17, 18] for example.

Nwaigwe, et al. [19-22], see also [23, 24] have suggested several numerical methods for nonlinear Fredholm equations, in particular they (see [19]) exploited the series expansion of the truncation error of the standard trapezoid rule, then applied one-sided finite difference approximations to derive a fourth-order numerical method for a second kind Fredholm integral equation. This present work is to extend that idea in [19] to the nonlinear first-order integro-differential equation in (1). Consequently, the details of our suggested numerical approximation of problem (1) are given in Section 2. Two numerical examples are given and used to assess the performance of the method in Section 3, and we give some concluding remarks in Section 4.

2. Numerical Algorithm

The numerical algorithm for approximating the solution of problem (1) is presented in this section. Select an $N \in \mathbb{Z}^+$, such that $N + 1 \geq 5$ is a number of points in $[a, b]$. Define a grid $\Omega_h = \left\{x_j = a + jh : j = 0, 1, \ldots, N, h := \frac{b-a}{N-1}\right\}.$

Define the operators:

$$\Delta_1^{\pm} f(x_i) = \pm(-3f(x_i) + 4f(x_i \pm h) - f(x_i \pm 2h)). \tag{2}$$

The following lemmas will be useful in the numerical formulation below.

Lemma 2.1 (Trapezoid Rule, see [25]). *Let* $f \in C^5[x_0, x_N]$, *and* h, x_j *be defined as above, for* $j = 0, 1, \ldots, N$ *and* $N \in \mathbb{Z}^+$. *Then*

$$\int_{x_0}^{x_N} f(x)dx = \frac{h}{2}\sum_{j=0}^{N-1}(f(x_j) + f(x_{j+1})) + \frac{h^2}{12}[f'(x_j) - f'(x_N)]$$

$$-\frac{h^4}{720}[f^{(3)}(x_0) - f^{(3)}(x_N)] + O(h^6). \tag{3}$$

Lemma 2.2 (See [26-27]). *Let* $0 < h \in \mathbb{R}$,

(i) *if* $f \in C^3(x_i - 2h, x_i)$, *then*

$$f'(x_i) = \frac{1}{2h}\Delta_1^- f(x_i) + O(h^2), \tag{4}$$

(ii) *if* $f \in C^3(x_i, x_i + 2h)$, *then*

$$f'(x_i) = \frac{1}{2h}\Delta_1^+ f(x_i) + O(h^2), \tag{5}$$

(iii) *if* $f \in C^5(x_i - 4h, x_i)$, *then*

$$f'(x_i) = \frac{1}{12h}(25f(x_i) - 48f(x_i - h) + 36(x_i - 2h)$$

$$-16(x_i - 3h) + 3f(x_i - 4h)) + O(h^4), \tag{6}$$

(iv) *if* $f \in C^5(x_i, x_i + 4h)$, *then*

$$f'(x_i) = \frac{1}{12h}(-25f(x_i) + 48f(x_i + h) - 36f(x_i + 2h)$$

$$+16f(x_i + 3h) - 3f(x_i + 4h)) + O(h^4), \tag{7}$$

and

(v) *if* $f \in C^5(x_i - 2h,\ x_i + 2h)$, *then*

$$f'(x_i) = \frac{1}{12h}\left(f(x_i - 2h) - 8f(x_i - h) + 8f(x_i + h) - f(x_i + 2h)\right) + O(h^4). \tag{8}$$

Equations (3)-(8) are used to derive the scheme below.

2.1. The numerical scheme

Collocate the integro-differential equation (1) at $x_i \in \Omega_h$, we get

$$\left.\frac{du}{dx}\right|_{x_i} - g(x_i) - f(x_i, u(x_i)) - \int_{x_0}^{x_N} k(x_i,\ y,\ u(y))dy = 0, \quad \forall x_i \in \Omega_h. \tag{9}$$

Using Lemmas 2.1 and 2.2, we proposed the approximations

$$\left.\frac{du}{dx}\right|_{x_i} = \frac{1}{12h}\begin{cases} -25u_i + 48u_{i+1} - 36u_{i+2} + 16u_{i+3} - 3u_{i+4},\ i = 1, \\ 25u_i - 48u_{i-1} + 36u_{i-2} - 16u_{i-3} + 3u_{i-4},\ i = N-1,\ N, \\ u_{i-2} - 8u_{i-h} + 8u_{i+h} - u_{i+2}, \text{ else}, \end{cases} \tag{10}$$

and

$$\int_{x_0}^{x_N} k(x_i,\ y,\ u(y))dy = I_i^h, \quad i = 1, 2, \ldots, N, \tag{11}$$

where

$$I_i^h = \sum_{j=0}^{N} \xi_j^N k(x_i,\ x_j,\ u_j) + \frac{h}{24}\left[-3k(x_i,\ x_0,\ u_0) + 4k(x_i,\ x_1,\ u_1) - 4k(x_i,\ x_2,\ u_2) - (k(x_i,\ x_N,\ u_N) - 4K(x_i,\ x_{N-1},\ u_{N-1}) + K(x_i,\ x_{N-2},\ u_{N-2})\right]. \tag{12}$$

Hence, we arrive at the following algorithm:

$$\left.\frac{du}{dx}\right|_{x_i} - g(x_i) - f(x_i, u(x_i)) - I_i^h = 0, \quad \text{for} \quad x_i \in \Omega_h, \tag{13}$$

where $\left.\frac{du}{dx}\right|_{x_i}$ and I_i^h are defined in (10) and (12) respectively, and ξ_j^N is defined as

$$\xi_j^N = h \begin{cases} 1/2 & \text{if } j = 0, N, \\ 1 & \text{if } 1 \le j < N, \\ 0 & \text{otherwise.} \end{cases}$$

The scheme (13) constitutes of a system of N equations in N unknowns. By including the initial condition of the problem we can summarize the scheme as follows:

$$\begin{cases} u_0 = u(x_0), \\ -25u_i + 48u_{i+1} - 36u_{i+2} + 16u_{i+3} - 3u_{i+4} - 12h\big(g(x_i) + f(x_i, u_i) \\ \qquad + I_i^h\big) = 0, \quad i = 1, \\ 25u_i - 48u_{i-1} + 36u_{i-2} - 16u_{i-3} + 3u_{i-4} - 12h\big(g(x_i) + f(x_i, u_i) \\ \qquad + I_i^h\big) = 0, \quad i = N-1, \ N, \\ u_{i-2} - 8u_{i-h} + 8u_{i+h} - u_{i+2} - 12h\big(g(x_i) + f(x_i, u_i) \\ \qquad + I_i^h\big) = 0, \quad 1 < i \le N-2. \end{cases} \tag{14}$$

This system is approximated using Newton-Raphson method and implemented in an in-house python code.

3. Numerical Experiments

In this section, we problem two examples of integro-differential equations, which have known close-form exact solution, for the purpose of verifying the accuracy of the method. The two examples are constructed via the method of manufactured solutions [20-21, 28-29]. In each example, the problem is solved on several grids with decreasing mesh sizes and the error of the numerical solution is computed in infinity norm, the experimental order of convergence is also recorded. The objectives are (i) to verify if the numerical solution computed by the proposed method converges to the exact solution, (ii) to observe the order of convergence of the method and if they agree with those of approximations used in deriving the method, and (iii) to observe stability numerically.

Example 3.1. Our first example is the following nonlinear integro-differential equation:

$$\frac{du(x)}{dx} = g(x) - \frac{1}{x^2u^2(x) + 3} + \int_0^1 \frac{xy^3 \cos(u(y))}{1 + x^2} dy, \quad x \in (0, 1), \quad u(0) = 0,$$

where

$$g(x) = -0.23168277 \frac{x}{1 + x^2} + \frac{1}{x^2 \tan^{-1}(0.5)^2 + 3} + \frac{0.5}{1 + 0.25x^2}.$$

The exact solution of this problem is

$$u(x) = \tan^{-1}(0.5x).$$

The solution of Example 3.1 is computed on a sequence of grids and the results are displayed in Table 1. It can be seen that the error of the numerical solution vanishes as the mesh size decreases. Also, the table shows that the observed (experimental) order of convergence is four which agrees with the approximation errors involved in the discritization of both the derivatives and integral. These results indicate that the method is convergent for this problem.

Further, Figures 1 and 2 display the plots of the exact and numerical solutions on different grids. The figures show that the numerical solution agrees with the exact solution even on a very coerce grid of ten points. Again, the plots show that there are no non-physical oscillations, which numerically indicate the stability of the method.

Example 3.2. The second example is

$$\frac{du(x)}{dx} = g(x) + \frac{xu^2(x)}{1 + u^2(x)} + \int_0^1 \frac{x^2y^2u^3(y)}{1 + u^4(y) - u^2(y)} dy, \quad x \in (0, 1), \quad u(0) = 0,$$

where

$$g(x) = \frac{x^2}{(1 + x^6)} \left(-x^5 + (1 + x^6)(-\pi\sqrt{3} + 162)/54 \right).$$

The exact solution of this problem is

$$u(x) = x^3.$$

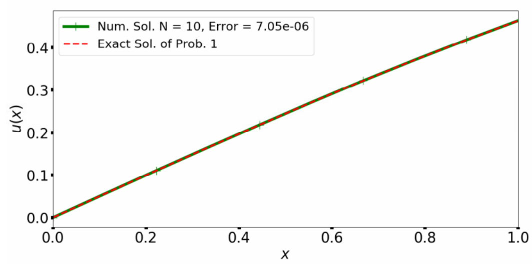
The results of the solution of Example 3.2 computed with the proposed method on a sequence of grids are displayed in Table 2. The table shows that the error (in infinity norm) decreases as the mesh size decreases. Like in Example 3.1 above, the results also show that the experimental order of convergence is four. Hence, we conclude that the method is also convergent for this problem. Figures 3 and 4, which display the plots of the exact and numerical solutions on different grids, show that the numerical solution computed with the proposed method agrees with the exact solution. The plots also show the absence of non-physical oscillations.

Table 1. Computed results for Example 1, where $N+1$ is the number of grid points, EOC is the Experimental Order of Convergence. The errors are computed in infinity norm

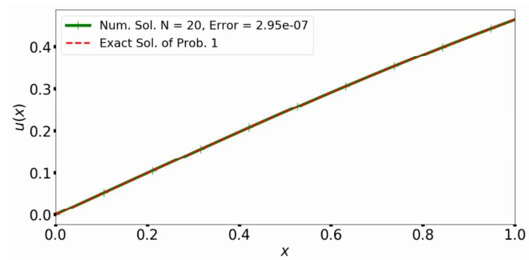
N	**Error (in Infinity Norm)**	**EOC**
6	6.43066587189267e-05	-
12	3.34534117701901e-06	4.2647428733
24	1.20519249149909e-07	4.7948178576
48	3.89512530563074e-09	4.9514500394
96	2.85725054727237e-10	3.7689702860
192	2.10523265486984e-11	3.7625759659
384	2.03176364621527e-12	3.3731751890

Table 2. Computed results for Problem 2, where $N+1$ is the number of grid points, EOC is the Experimental Order of Convergence. The errors are computed in infinity norm

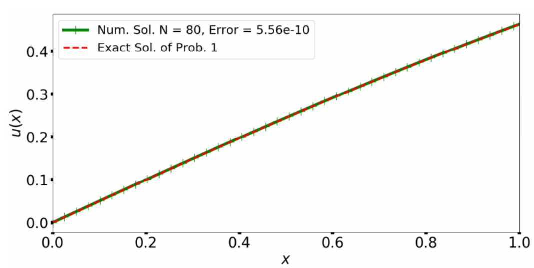
N	**Error (in Infinity Norm)**	**EOC**
10	0.000856277542524153	-
20	9.15957652634081e-05	3.224725683573
40	5.28402751687196e-06	4.115571015276
80	2.78801202235890e-07	4.244329292054
160	1.53948808145898e-08	4.178714178996
320	8.95607366047102e-10	4.103440494135
640	5.69509994718942e-11	3.975073327779



(a) Plot of numerical and exact solution of Example 1 with 10 mesh points



(b) Plot of numerical and exact solution of Example 1 with 20 mesh points



(c) Plot of numerical and exact solution of Example 1 with 80 mesh points

Figure 1. Plots of exact and numerical solutions of Problem 1.

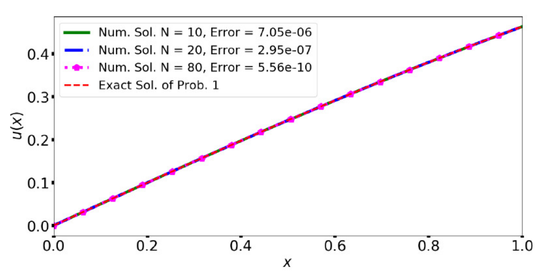
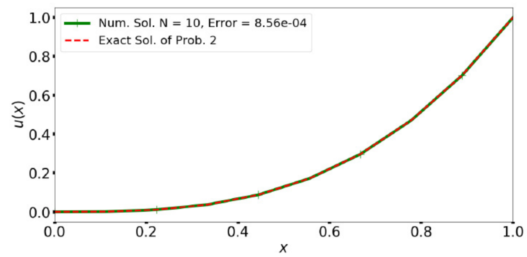
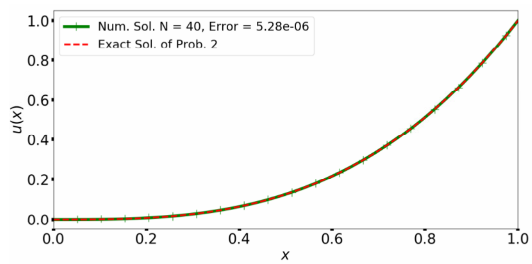


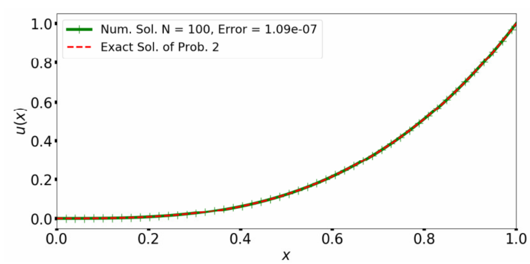
Figure 2. Plot of exact and numerical solutions of Example 1 on different grids.



(a) Plot of numerical and exact solution of Example 2 with 10 mesh points



(b) Plot of numerical and exact solution of Example 2 with 40 mesh points



(c) Plot of numerical and exact solution of Example 2 with 100 mesh points

Figure 3. Plots of exact and numerical solutions of Problem 2.

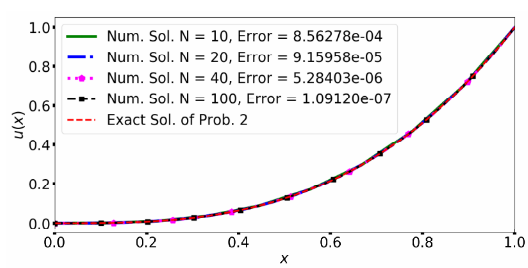


Figure 4. Plot of exact and numerical solutions of Example 2 on different grids.

4. Conclusion

A high-order numerical scheme, (14), is proposed for Fredholm-type first-order integro-differential equations. The simplicity of the derivation is that all the approximations are based on fundamental concepts of numerical analysis which can all be derived from Taylor's theorem. The proposed scheme which is in the form of a nonlinear algebraic system is implemented in python and two examples are used to assess the performance of the algorithm. The numerical results, which are presented using tables and figures, show that

(i) the numerical solution, computed with the proposed method on a sequence of grids, converges to the exact solution,

(ii) the method has high-order (fourth order) of accuracy, and

(iii) it produces no non-physical oscillations, even on a very coerce mesh; this gives an indication of the stability of the method.

References

[1] Vangipuram Lakshmikantham, Theory of Integro-differential Equations, Volume 1, CRC Press, 1995.

[2] R. C. MacCamy, An integro-differential equation with application in heat flow, Quart. Appl. Math. 35(1) (1977), 1-19.

[3] Ravi P. Agarwal, Bruno de Andrade and Giovana Siracusa, On fractional integro-differential equations with state-dependent delay, Comput. Math. Appl. 62(3) (2011), 1143-1149.

[4] R. C. MacCamy, A model for one-dimensional, nonlinear viscoelasticity, Quart. Appl. Math. 35(1) (1977), 21-33.

[5] Alexander Keimer and Lukas Pug, Modeling Infectious Diseases using Integro-differential Equations: Optimal Control Strategies for Policy Decisions and Applications in Covid-19, Res Gate, 10, 2020.

[6] Jamshad Ahmad, Angbeen Iqbal and Qazi Mahmood Ul Hassan, Study of nonlinear fuzzy integro-differential equations using mathematical methods and applications, Internat. J. Fuzzy Logic Intell. Syst. 21(1) (2021), 76-85.

[7] Mutaz Mohammad and Alexander Trounev, Implicit Riesz wavelets based-method for solving singular fractional integro-differential equations with applications to hematopoietic stem cell modeling, Chaos, Solitons & Fractals 138 (2020), 109991.

[8] Luisa Arlotti, Elena De Angelis, Luisa Fermo, Miroslaw Lachowicz and Nicola Bellomo, On a class of integro-differential equations modeling complex systems with nonlinear interactions, Appl. Math. Lett. 25(3) (2012), 490-495.

[9] Harsh Vardhan Jain and Helen M Byrne, Qualitative analysis of an integro-differential equation model of periodic chemotherapy, Appl. Math. Lett. 25(12) (2012), 2132-2136.

[10] Fawziah Al-Saar and Kirtiwant Ghadle, Solving nonlinear Fredholm integro-differential equations via modifications of some numerical methods, Adv. Theo. Nonlin. Anal. Appl. 5(2) (2021), 260-276.

[11] D. Shirani, Kajani M. Tavassoli and S. Salahshour, Numerical solution of system of fredholm-volterra integro-differential equations using Legendre polynomials, Filomat 36(5) (2022), 1685-1697.

[12] Mohamed El-Gamel and Ola Mohamed, Nonlinear second order systems of Fredholm integro-differential equations, SeMA J. 79(2) (2022), 383-396.

[13] Salwa A. Mohamed, Norhan A. Mohamed and Sarah I. Abo-Hashem, A novel differential-integral quadrature method for the solution of nonlinear integro-differential equations, Math. Methods Appl. Sci. 44(18) (2021), 13945-13967.

[14] Mohammed Charif Bounaya, Samir Lemita, Mourad Ghiat and Mo-hamed Zine Aissaoui, On a nonlinear integro-differential equation of Fredholm type, Internat. J. Comput. Sci. Math. 13(2) (2021), 194-205.

[15] D. I. Lanlege, F. M. Edibo and Momoh Sheidu Omeiza, Solution of Fredholm integro-differential equation by variational iteration method, Fudma J. Sci. 7(2) (2023), 1-8.

[16] Taherh Tahernezhad and Reza Jalilian, Exponential spline for the numerical solutions of linear Fredholm integro-differential equations, Adv. Diff. Equat. 2020 (2020), 1-15.

[17] Chinedu Nwaigwe and Deborah Ngochinma Benedict, Generalized banach fixed-point theorem and numerical discretization for nonlinear Volterra-Fredholm equations, J. Comput. Appl. Math. 425 (2023), 115019.

[18] Sohrab Bazm, Pedro Lima and Somayeh Nemati, Analysis of the Euler and trapezoidal discretization methods for the numerical solution of nonlinear functional Volterra integral equations of Urysohn type, J. Comput. Appl. Math. 398 (2021), 113628.

[19] Chinedu Nwaigwe, Azubuike Weli and Dang Ngoc Hoang Thanh, Fourth-order trapezoid algorithm with four iterative schemes for nonlinear integral equations, Lobachevskii J. Math. 44(7) (2023), 2822-2837.

[20] Chinedu Nwaigwe and Azubuike Weli, Ishikawa-collocation method for nonlinear Fredholm equations with non-separable kernels, J. Adv. Math. Comput. Sci. 38(3) (2023), 1-11.

[21] Azubuike Weli and Chinedu Nwaigwe, Computational analysis of two numerical solvers for functional Fredholm equations, Internat. J. Math. Comput. Meth. 8 (2023), 1-8.

[22] Azubuike Weli and Chinedu Nwaigwe, On computational efficiency of nine iterative algorithms for Hammerstein equations, Asian Res. J. Math. 19(5) (2023), 1-15.

[23] Chinedu Nwaigwe and Sanda Micula, Fast and accurate numerical algorithm with performance assessment for nonlinear functional Volterra equations, Fract. Fraction. 7(4) (2023), 333.

[24] Chinedu Nwaigwe, Solvability and approximation of nonlinear functional mixed Volterra-Fredholm equation in Banach space, J. Integ. Equat. Appl. 34(4) (2022), 489-500.

[25] Bengt Fornberg, Improving the accuracy of the trapezoidal rule, SIAM Revi. 63(1) (2021), 167-180.

[26] Richard L. Burden, Numerical Analysis, 2011.

[27] E. Atkinson Kendall, et al., An Introduction to Numerical Analysis, John Wiley & Sons, Inc., New York, USA, 1989, pp. 37-45.

[28] Azubuike Weli and Chinedu Nwaigwe, Numerical analyses of channel flow with velocity-dependent suction and nonlinear heat source, J. Interdiscipl. Math. 23(5) (2020), 987-1008.

[29] Chinedu Nwaigwe and Azubuike Weli, Analysis of two finite difference schemes for a channel flow problem, Asian Res. J. Math. 15 (2019), 1-14.