A NOTE ON THE VERTEX-DISTINGUISHING EDGE COLORING OF $P_m \lor K_n$ AND $C_m \lor K_n$

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Abstract

In this paper, we obtain the Vertex-distinguishing Edge Chromatic Number of $P_m \lor K_n$ and $C_m \lor K_n$.

1. Introduction

The problem which is due to computer science [1-6] about Vertex-distinguishing Edge Coloring of $G$ is a widely applicable and extremely difficult problem. In [7] introduced the Vertex-distinguishing Edge Coloring of graph, and give the relevant conjecture.

Definition 1 [8-10]. $G$ is a simple graph and $k$ is a positive integer, if it exists a mapping of $f$, and satisfied with $f(e) \neq f(e')$ for adjacent edge $e, e' \in E(G)$, then $f$ is called a Proper Edge Coloring of $G$, is abbreviated $k$-PEC of $G$, and

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\[ \chi'(G) = \min \{ k \mid k \text{PEC} \} \]

is called the Edge Chromatic Number of \( G \).

**Definition 2** [1-6]. For the proper edge coloring \( f \) of simple graph, if it is satisfied with \( C(u) \neq C(v) \) for \( V(G) (u \neq v) \), where \( C(U) = \{ f(uv) \mid uv \in E(G) \} \), then \( f \) is called the Vertex-distinguishing Edge Coloring, is abbreviated \( k \)-VDEC of \( G \), and

\[ \chi'_{vd} = \min \{ k \mid k \text{VDEC} \} \]

is called the Vertex-distinguishing Edge Chromatic Number of \( G \).

**Definition 3.** For a graph \( G \), let \( n_i \) be the vertex number of the vertices of degree \( i \), we call

\[ \mu(G) = \max \left\{ \min \left\{ \lambda \left( \frac{n_i}{\delta} \right) \geq n_i, \delta \leq i \leq \Delta \right\} \right\} \]

the Combinatorial Degree of \( G \), where \( \delta \) and \( \Delta \) are the minimal and maximal degree of \( G \), respectively.

**Conjecture** [1-5]. For a connected graph \( G \) of order not less than 3, then

\[ \mu(G) \leq \chi'_{vd} \leq \mu(G) + 1. \]

Note that the left side of the inequality is obviously true.

Let \( G \) and \( H \) are two simple graphs, the joint graph of \( G \) and \( H \), denote by \( G \cup H \), is obtained from the disjoint union of \( G \) and \( H \) by making all of \( V(G) \) adjacent to all of \( V(H) \).

Because \( P_1 \cup K_n = K_{n+1} \) and \( P_2 \cup K_n = K_{n+2} \) has been discussed in another paper, we will consider the general case \( P_m \cup K_n \) and \( C_m \cup K_n \). The terms and signs we use in this paper but not denoted can be found in [8-10].

**Lemma 1.** Let \( m \geq 3 \) and \( n \geq 4 \), \( \mu(P_m \cup K_n) = m + n \).

**Proof.** For \( m = 3 \) and \( n = 3 \), we can compute that

\[ \max \left\{ \min \left\{ \theta \left( \frac{\theta}{6} \right) \geq 2 \right\} \text{ and } \min \left\{ \theta \left( \frac{\theta}{7} \right) \geq 6 \right\} \right\} = 8. \]
For $n \geq 4$ and $m + n \neq 8$, we get that

$$\max\left\{ \min\left\{ \frac{\theta}{n+1} \geq 2 \right\}, \min\left\{ \frac{\theta}{n+2} \geq m - 2 \right\} \text{ and } \min\left\{ \frac{\theta}{m+n-1} \geq n \right\} \right\}$$

$$= m + n.$$  

Hence, the proof is finished.

**Lemma 2 [5].** For a complete graph $K_n$, then

$$\chi'_{vd}(K_n) = \begin{cases} n + 1, & \text{for } n \equiv 0 \pmod{2}; \\ n, & \text{for } n \equiv 1 \pmod{2}. \end{cases}$$

**Lemma 3.** If $m \geq 3$ and $n \geq 4$, then

$$\mu(C_m \lor K_n) = m + n.$$  

**Proof.** We have that

$$\mu(C_m \lor K_n)$$

$$= \max\left\{ \min\left\{ \frac{\theta}{n+1} \geq m \right\} \text{ and } \min\left\{ \frac{\theta}{m+n-1} \geq n \right\} \right\}$$

$$= m + n.$$  

2. Results about $P_m \lor K_n$

**Theorem 2.1.** If $m + n \neq 3$, then

$$\chi'_{vd}(P_m \lor K_n) = \begin{cases} n + 1, & m = 1, n \equiv 0 \pmod{2}; \\ n + 2, & m = 1, n \equiv 1 \pmod{2}; \\ n + 3, & m = 1, n \equiv 0 \pmod{2}. \end{cases}$$

**Proof.** When $m = 1, 2$, we can get $P_m \lor K_n = K_{m+n}$ from [5], the conclusion is true.

**Theorem 2.2.** If $m \geq 3$ and $n \geq 4$, then

$$\chi'_{vd}(P_m \lor K_n) = m + n.$$
Proof. Let the path $P_m = u_1u_2 \cdots u_m$ and $V(K) = \{u_{m+1}, u_{m+2}, \ldots, u_{m+n}\}$ and $C = \{1, 2, \ldots, m + n - 1, 0\}$. From Lemma 2, we only need to prove that there exists a $(m + n)$-VDEC of $P_m \vee K_n$. Hence, we can make a proper edge coloring $f$ of $P_m \vee K_n$ as:

$$f(u_i u_j) = i + j - 1 \pmod{m + n} \quad \text{for} \quad 1 \leq i \leq n \quad \text{and} \quad m + 1 \leq j \leq m + n,$$

and

$$f(u_m i u_{m+j}) = 2m + i + j - 2 \pmod{m + n} \quad \text{for} \quad i \leq i, \quad j \leq n.$$

Let the color subtractive set $\overline{C}(u) = C \setminus C(u)$ for $u \in V(P_m \vee K_n)$.

**Case 1.** If $m > n \geq 4$, $f(u_i u_{i+1}) = i$ for $1 \leq i \leq n$; we can compute that

- $\overline{C}(v_i) = \{2(i - 1)\}$, for $1 \leq i \leq n$;
- $C(u_1) = \{1, n, n + 1, \ldots, 2n - 1\}$;
- $C(u_m) = \{m - 1, m + n - 1, 0, \ldots, n - 2\}$;
- $C(u_i) = \{i - 1, i, n + i - 1, n + i, \ldots, 2n + i - 2\} \pmod{m + n}$, for $2 \leq i \leq m - 1$.

Thus $f$ is a $(m + n)$-VDEC of $P_m \vee K_n$. This proves that the result is true.

**Case 2.** If $m = n$, $f(u_i u_{i+1}) = i$ for $1 \leq i \leq n - 1$, there are

- $\overline{C}(v_i) = \{2(i - 1)\}$, for $1 \leq i \leq n$;
- $C(u_1) = \{1, n, n + 1, \ldots, 2n - 1\}$;
- $C(u_m) = \{n - 1, 2n - 1, 0, \ldots, n - 2\}$;
- $C(u_i) = \{i - 1, i, n + i - 1, n + i, \ldots, 2n + i - 2\} \pmod{2n}$, for $2 \leq i \leq n - 1$.

Hence, $f$ is $(m + n)$-VDEC of $P_m \vee K_n$.

**Case 3.** If $n > m$, there are

$$f(u_i u_{i+1}) = n - m + i,$$ for $1 \leq i \leq m - 1,$
we get that
\[
\overline{C}(v_i) = \{2(i - 1)\}, \quad \text{for } 1 \leq i \leq \frac{m + n}{2};
\]
\[
\overline{C}(v_i) = \{2i - m - n + 1\}, \quad \text{for } \frac{m + n}{2} \leq i \leq n.
\]

For \(m + n \equiv 1(\text{mod } 2)\), there have
\[
\overline{C}(v_i) = \{2(i - 1)\}, \quad \text{for } 1 \leq i \leq \frac{m + n + 1}{2};
\]
\[
\overline{C}(v_i) = \{2i - m - n + 1\}, \quad \text{for } \frac{m + n + 1}{2} + 1 \leq i \leq n,
\]
\[
C(u_1) = \{n - m + 1, n, n + 1, \ldots, 2n - 1\} (\text{mod } m + n);
\]
\[
C(u_m) = \{n - 1, m + n - 1, 0, 1, \ldots, n - 2\};
\]
\[
C(u_i) = \{n - m + i, n - m + i + 1, n + i - 1, \ldots, 2n + i - 2\} (\text{mod } m + n),
\]
for \(1 \leq i \leq m - 1\).

Therefore, \(f\) is a \((m + n)\)-VDEC of \(P_m \vee K_n\). The proof is finished.

3. Results about \(C_m \vee K_n\)

**Theorem 3.1.** If \(n > 1\), then
\[
\chi'_{vd}(C_3 \vee K_n) = \begin{cases} 
 n + 4, & n = 1(\text{mod } 2); \\
 n + 3, & n = 0(\text{mod } 2).
\end{cases}
\]

**Proof.** Because of \(C_3 \vee K_n = K_{n+3}\), the result is true we know by [5].

**Theorem 3.2.** If \(m \geq 4\) and \(n \geq 4\), then
\[
\chi'_{vd}(C_m \vee K_n) = m + n.
\]

**Proof.** By Lemma 2, the inequality \(\chi'_{vd}(C_3 \vee K_n) \geq \mu(C_3 \vee K_n)\) is obvious, so
we only need to prove that $C_3 \lor K_n$ has a mapping $(m+n)$-VDEC only. For convenient, we let that

$$C_m = u_1u_2\ldots u_{m+1},$$
$$V(K_n) = \{v_i \mid i = 1, 2, \ldots, n\};$$
$$C = \{1, 2, \ldots, m + n - 1, 0\},$$
$$\overline{C}(v) = C \setminus C(v),$$
$$u_i = v_{n+i}, \quad i = 1, 2, \ldots, m.$$

**Case 1.** If $m > n$, we make a coloring function $f$ as:

$$f(v_i v_j) = i + j - 2(\text{mod } m + n),$$

for $i = 1, 2, \ldots, n; \quad j = i + 1, i + 2, \ldots, m + n$ and $f(u_{m+1}) = i, \quad i = 1, 2, \ldots, m - 1; \quad$ and $f(u_{m}u_1) = n - 1.$

Therefore, we can get that

$$\overline{C}(v_i) = \{2(i - 1)\}, \quad \text{for } 1 \leq i \leq n;$$

$$C(u_1) = \{1, n - 1, n, \ldots, 2n - 1\};$$
$$C(u_m) = \{m - 1, m + n - 1, 0, 1, \ldots, n - 1\};$$
$$C(u_i) = \{i - 1, i, n + i - 1, \ldots, 2n + i - 2\} (\text{mod } m + n), \quad \text{for } 2 \leq i \leq m - 1.$$

This proves that $f$ is a $(m+n)$-VDEC of $C_m \lor K_n$.

**Case 2.** If $m = n$, we make $f$ as:

$$f(v_i v_j) = i + j - 2(\text{mod } 2n), \quad i = 1, 2, \ldots, n; \quad j = i + 1, i + 2, \ldots, 2n$$

and

$$f(u_{m+1}) = i + 1, \quad i = 1, 2, \ldots, n - 1;$$

and $f(u_{m}u_1) = n - 1.$
Then, we still have that
\[ C(v_i) = \{2(i-1)\}, \quad i = 1, 2, \ldots, n; \]
\[ C(u_i) = \{2, n-1, n, \ldots, 2n-1\}; \]
\[ C(u_m) = \{n-1, n, 2n-1, 0, 1, \ldots, n-2\}; \]
\[ C(u_i) = \{i, i+1, n+i-1, \ldots, 2n+i-2\} \pmod{2n}, \quad i = 2, 3, \ldots, n-1. \]
That means that \( f \) is a \((2n)\)-VDES of \( C_m \vee K_n \).

**Case 3.** If \( n > m \), we let \( f \) as:
\[ f(v_i,v_j) = i + j - 2 \pmod{m+n}, \quad i = 1, 2, \ldots, n; \quad j = i + 1, i + 2, \ldots, m+n \]
and
\[ f(u_iu_{i+1}) = n - m + i, \quad i = 1, 2, \ldots, m-2; \]
and \( f(u_{m-1}u_m) = n \) and \( f(u_mu_1) = n-1. \)

Then, if \( m+n \equiv 0 \pmod{2} \), we can see that
\[ \overline{C}(v_i) = \{2(i-1)\}, \quad i = 1, 2, \ldots, \frac{m+n}{2}; \]
\[ \overline{C}(v_i) = \{2i - (m+n) - 1\}, \quad i = \frac{m+n}{2} + 1, \frac{m+n}{2} + 2, \ldots, n; \]
\[ C(u_1) = \{n - m + 1, n - 1, n, \ldots, n - m - 1\}; \]
\[ C(u_{m-1}) = \{n - 2, n, m + n - 2, m + n - 1, 0, 1, \ldots, n - 3\}; \]
\[ C(u_m) = \{n - 1, n, m + n - 1, 0, \ldots, n - 2\}; \]
and
\[ C(u_i) = \{n - m + i, n - m + i + 1, n + i - 1, \ldots, n - m + i - 2\} \pmod{m+n}, \quad i = 2, 3, \ldots, m-2. \]
If \( m + n \equiv 1 \pmod{2} \), we can compute

\[
\overline{C}(v_i) = \{2(i - 1)\}, \quad i = 1, 2, \ldots, \frac{m + n + 1}{2};
\]

\[
\overline{C}(v_i) = \{2i - m - n\}, \quad i = \frac{m + n + 1}{2} + 1, \frac{m + n + 1}{2} + 2, \ldots, n;
\]

\[C(u_1) = \{n - 1, n - m + 1, n, n + 1, \ldots, n - m - 1\};\]
\[C(u_{m-n}) = \{n - 2, n, m + n - 2, m + n - 1, 0, 1, \ldots, n - 3\};\]
\[C(u_{m}) = \{n - 1, n, m + n - 1, 0, \ldots, n - 2\};\]
\[C(u_i) = \{n - m + i - 1, n - m + i, n + i - 1, \ldots, n - m + i - 2\}(\mod{n + m}), \quad i = 2, 3, \ldots, m - 2.\]

We have proved that \( f \) is a \((m + n)\)-VDEC of \( C_m \vee K_n \).

The proof is completed.

References

